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PREFACE

This volume brings forth a set of papers presented at the

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The workshop gathered more than 40 participants coming from Western European countries (Austria, Belgium, Finland, Germany, Netherland, Yugoslavia), Eastern European countries (Bulgaria, Czechoslovakia, Hungary, Poland, USSR) and non-European countries such as Canada, China and Iran.

The papers included in the volume are arranged in alphabetical order.

I wish to thank all contributors for their valuable papers and an outstanding cooperation in the editorial perfect.

Brussel, April 1991.

Prof. Marc Roubens
IFSA-EC and EURO-WGFS
President

THIRD JOINT IFSA–EC AND EURO–WG WORKSHOP ON FUZZY SETS

Visegrád, Hungary, 11–13 December, 1990

TOWARDS A UNIFIED FUZZY SETS THEORY

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ARITHMETICS WITH QUALITATIVE VALUES AND FUZZY SET THEORY

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Abstract: Arithmetics with qualitative values based on fuzzy set theory is introduced. It can be used for qualitative reasoning tasks when at least vague and subjective knowledge about variables magnitudes or their relative magnitudes is known. Simple three element fuzzy vectors are used to model qualitative values and algebraic operations are defined in terms of manipulations with their membership functions. Obtained results coincide quite satisfactory with our common sense.

Keywords: Qualitative reasoning, fuzzy sets, qualitative arithmetics, common-sense arithmetics

1. INTRODUCTION

A model as an intellectual construction is very important in human reasoning about the world. One of reasons why human behavior is (relatively) successful is certainly the propensity of human brain to create qualitative models of the world and to use them in qualitative reasoning about the world. It is not surprising that a lot of attempts have been made to understand and to apply qualitative reasoning to different areas of scientific thinking.

The most simple quantity space which can be used in qualitative reasoning has just two values: "existence" and "nonexistence-zero" of interactions, features or values. Splitting the value "existence" into two parts, emphasizing opposite meanings, the well known three value sign quantity space with "positive", "negative" and "zero" values can be established. In 1946, Heider [1] used this three value quantity space to analyze the complex psychological behavior of people in a small group. After him, qualitative reasoning with sign values positive, negative and zero has found interesting and useful application in areas where organization and/or interactions are complex, specially in sociology [2,3] biology and environmental sciences [3], chemistry [4], but also in technical sciences [5].

The main disadvantage of qualitative modelling with sign values is its inherent ambiguity. In lots of qualitative inferences with sign values plus and minus the result could not be found, because addition of these values is not defined. To solve this problem a number of approaches have been proposed, from subdividing the positive and negative into more detailed subintervals by introducing new landmarks on the real line, to the introduction of reasoning with order magnitudes.

Here an alternative approach will be proposed, an approach based on fuzzy set theory. Qualitative values are defined by fuzzy sets and arithmetics with qualitative values by manipulators with fuzzy set membership functions. Our task is not to give answers to all open questions, but to show how ideas from fuzzy set theory could be used to define and solve some problems of qualitative reasoning.

2. DEFINITION OF QUALITATIVE VALUES WITH FUZZY SETS

In 1964, Hararry et al. [2] introduced the sign arithmetic and defined addition $\oplus$ and multiplication $\otimes$ of sign values with Cayley tables using common sense about calculations with signs. For example $+ \oplus + = +$ or $+ \otimes - = -$. Instead of three values quantity space they used a four valued quantity space $Q = \{+, -, 0, ?\}$, where ? means 'ambivalent', 'ambigipis', 'indeterminate' or 'any sign value'. Without ? it was not possible to define addition as a closed arithmetic operation.

More recently Struss [6] introduced the interval arithmetic in qualitative reasoning as a more general approach than the sign arithmetic, because it allows to use more complicated quantity space. The core of his approach is representation of qualitative values with intervals, so the quantity space is the set of open and closed intervals defined by the set of landmarks B. For sign quantity space the set of landmarks is $B = -\infty,\ 0,\ +\infty$, so $-$, $+$, 0 and ? are defined as $(-\infty, 0)$, $(0, +\infty)$, $[0, 0]$ and $(-\infty, +\infty)$ respectively. For addition and multiplication well known interval arithmetics is used.

If the landmarks are introduced a lots of problems arise [6], as for example the problem of finiteness of the quantity space, the problems of definition of operation subtraction etc. Some of these problems could be resolved by modifying interval arithmetics by restricting the quantity space to a set of elementary intervals on, but then we have another problems, typically the operation addition is not any more associative.

To extend the quantity space instead of introducing new landmarks the set of the same landmarks as in the sign quantity space $B = \{-\infty, 0, \infty\}$ could be used and qualitative values could be defined by discrete fuzzy sets whose support set is the set B. The value of the membership function could be seen as a plausibility degree to which the qualitative value is more or less "close" to the appropriate landmark. For example the membership function of the big negative qualitative value could be expressed using the usual notation from the fuzzy set theory as $1/-\infty$, $+0.2/0$, $+0/+\infty$, or more conveniently in the form of *fuzzy vector* [1 0.2 0] where the first element corresponds to the support element $-\infty$, the second element to 0 and the third to $+\infty$. Similarly a qualitative value with the smaller magnitude, but still negative, could be expressed as [1 0.5 0], and approximately the same positive qualitative value as [0 0.5 1]. Note that [0 0.5 1] represent the qualitative value smaller than [0 0.2 1]. One restriction to the representation of qualitative values with fuzzy sets is introduced. That is that the fuzzy set must be normalized fuzzy set, so at least one of the elements of the fuzzy vector must be equal to 1. The reason is because the interpretation of the results is easier.

As in all other cases when the fuzzy sets are used to model the real world, the question is how to assign membership values. One solution could be subjective, context dependent assignment based on human observation. Qualitative reasoning is usually connected with changes of variables and humans can easily judge (if the time constants are acceptable to human perception) is the change big, very big, small or maybe more or less small. Now a theory of linguistic modelling based on fuzzy set theory could be used. Only the membership functions of primary linguistic terms have to be defined in advance, as for example "positive big" could be defined as [0 0.2 1] or "negative medium" as [1 0.5 0]. For hedges standard operations from linguistic modelling theory could be used. Typical example is to use 2nd power for "very", so "very positive big" could be represented with [0 0.4 1] if 'positive big' is [0 0.2 1].

3. ARITHMETICS WITH FUZZY QUALITATIVE VALUES

Arithmetic operations are defined using well developed fuzzy arithmetics but additionaly requirements are established. These requirements could be divided into three groups:

a) *Formal arithmetic requirements.* They include algebraic laws as associativity, commutativity, etc.

b) *Common knowledge requirements.* Qualitative arithmetics has to coincide, as much as possible, with qualitative interpretation of real line arithmetic. Typical example is that the result of multiplication of the value higher then 1 and the value smaller than 1 must be somewhere between them.

c) *Implementation requirements.* Computation procedure must be simple, interpretation and presentation of results easy, requirements for data storage acceptable, etc.

Let us start with the operation addition. According to Zadeh's extension principle the sum of two non-interacting fuzzy sets with membership functions X^* and $Y^*(y)$ is a fuzzy set U^* with membership function $U^*(u)$

$$U^*(u) = \bigvee_{x+y=u} (X^*(x) \wedge Y^*(y)) \tag{1}$$

where $\vee$ is max and $\wedge$ is min. In our case all fuzzy sets including the resulting fuzzy set have the same support set $\{-\infty, 0, \infty\}$. According to our knowledge about extended algebra of limits the following assumptions hold:

a) $(-\infty) + (-\infty)$ and $(-\infty) + 0$ give $-\infty$

b) $(\infty) + (\infty)$ and $(\infty) + 0$ give ∞, and

c) 0 can be obtained only as a result of $0 + 0$, because $(-\infty) + (\infty)$ is indeterminate case.

The addition $\oplus$ of two fuzzy vectors $\underline{X}^* = [x_1\ x_2\ x_3]$ and $\underline{Y}^* = [y_1\ y_2\ y_3]$ could be defined, using the extension principle, as

$$\underline{U}^* = \underline{X}^* \oplus \underline{Y}^* = [u_1\ u_2\ u_3] \tag{2}$$

where $u_1 = u_1'/u,\ u_2 = u_2'/u,\ u_3 = u_3'/u,\ u = u_1' \vee u_2' \vee u_3'$ and

$$u_1' = (x_1 \wedge y_1) \vee (x_1 \wedge y_2) \vee (x_2 \wedge y_1) \tag{3}$$

$$u_2' = x_2 \wedge y_2 \tag{4}$$

$$u_3' = (x_2 \wedge y_3) \vee (x_3 \wedge y_2) \vee (x_3 \wedge y_3) \tag{5}$$

The resulting fuzzy vector is normalized because that was the restriction introduced at the begining. The addition defined by such a way is a commutative and associative operation and [0 1 0] is the addition identity (neutral element).

Multiplication $\otimes$ could be defined using the same extension principle and knowledge that $\infty \cdot \infty$ and $(-\infty) \cdot (-\infty)$ give ∞, $(-\infty) \cdot \infty)$ gives $-\infty$, $0 \cdot 0$ gives 0 and $(-\infty) \cdot 0$ and $\infty \cdot 0$ are indeterminate cases, but we have applied a slightly modified definition replacing the min operator with ordinary product, so

$$\underline{V}^* = \underline{X}^* \otimes \underline{Y}^* = [v_1\ v_2\ v_3] \tag{6}$$

where $v_1 = v_1'/v,\ v_2 = v_2'/v,\ v_3 = v_3'/v,\ v = v_1' \vee v_2' \vee v_3'$ and

$$v_1' = (x_1 \cdot y_3) \vee (x_3 \cdot y_1) \tag{7}$$

$$v_2' = x_1 \cdot y_2 \tag{8}$$

$$v_3' = (x_1 \cdot y_1) \vee (x_3 \cdot y_3) \tag{9}$$

This modification was introduced because the results correspond better to our common knowledge about multiplying quantities. Let us explain that more deeply.

If we multiply two big positive quantities the result usually has bigger order of magnitude than the biggest of quantities used in multiplication. For addition that is not the case. The sum normally have the same order of magnitude as the biggest quantity. If the operation min is used to define multiplication this property is not emphasized, as it is for the equations (7) to (9).

The multiplication is also commutative and associative operation but unfortunately it is not distributive over addition. Both multiplicative zero [0 1 0] and multiplicative identity [0 1 1] exist. Multiplicative zero [0 1 0] is the same as the addition identity and it corresponds to the real line element zero-0, which is also identity element in ordinary addition and zero element in ordinary multiplication on the real line. Because of this correspondence it will be interesting to analyze the similarities between the multiplicative identity [0 1 0] and its corresponding real line element, the unity-1, which has the same properties according to ordinary real line multiplication.

The fuzzy sets which represent qualitative values must be normalized. This means that for example strictly positive values could be represented on two ways, with fuzzy vectors [0 x 1] and [0 1 x] where $x \leq 1$. The fuzzy vector [0 1 1] is a boundary case for both representations. As it corresponds to the number 1, it seems natural to use [0 x 1] for representing qualitative values bigger than 1 and to use [0 1 x] for representing qualitative values smaller than 1.

Let us test this approach by introducing some common knowledge laws about multiplication:

a) The product of two positive values bigger than 1 is bigger than the biggest of them.
Example: $[0\ 0.2\ 1] \cdot [0\ 0.5\ 1] = [0\ 0.1\ 1]$.

b) The product of two positive values smaller than 1 is smaller than the smallest of them.
Example: $[0\ 1\ 0.4] \cdot [0\ 1\ 0.2] = [0\ 1\ 0.08]$.

c) The product of positive value bigger than 1 and a positive values smaller than 1 must be somewhere between them.
Example: $[0\ 0.2\ 1] \cdot [0\ 1\ 0.5] = [0\ 0.4\ 1]$.

Obtained results completely correspond to our common knowledge. The situation is similar for the negative qualitative values or qualitative values with mixed signs. The negative unity (-1) is [1 1 0], so the negative qualitative values with magnitudes bigger than 1 are modeled with [1 x 0] and with magnitudes smaller than 1 with [x 1 0] where $x \leq 1$.

Let us now return back to addition and analyze the addition of positive and negative quantities. Two examples are $[0\ 0.2\ 1] \oplus [1\ 0.2\ 0] = [1\ 1\ 1]$ and $[0\ 0.2\ 1] \oplus [1\ 0.4\ 0] = [0.5\ 0.5\ 1]$. The resulting fuzzy vectors have all three element s different from 0. This means ambiguity, but with adequate interpretation, a non-ambiguous result could be obtained. Interpretation has to be defined taking into account assumptions as:

a) The result of addition of a big positive qualitative value and a negative qualitative value of the similar magnitude must be close to zero;

b) The result of addition of a big positive qualitative value and a smaller negative qualitative value is a positive qualitative value, but not so big as the first one.

The first idea is to solve this problem using the real number interpretation of the resulting fuzzy sets. The center of gravity method could be the appropriate procedure for this approach. Another idea which is more complicated, but more interesting consists of transforming the fuzzy sets with all three elements different from 0 into strictly positive or strictly negative qualitative values. This means that fuzzy set where all three elements are diferent from zero has to be transformed into fuzzy set where one of the elements (first or third) is equal to 0. In software QUANA which is shortly described in Chapter 4 one such method is applied.

Another two important binary algebraic operations are subtraction and division. Williams has noticed [8] that the subtraction could not be defined as the inverse of addition, but subtraction is related to addition in a manner similar to ordinary algebra:

$$\ominus \underline{Y}^* = [1\ 1\ 0] \otimes \underline{Y}^* \tag{10}$$

$$\underline{X}^* \ominus \underline{Y}^* = \oplus([1\ 1\ 0] \otimes \underline{Y}^* \tag{11}$$

where $[1\ 1\ 0]$ is a fuzzy representation of negative unity (-1). It is important to emphasize that properties as

$$\ominus(\ominus \underline{X}^*) = \underline{X}^* \tag{12}$$

$$\ominus(\underline{X}^* \oplus \underline{Y}^*) = \ominus \underline{X}^* \ \ominus \underline{Y}^* \tag{13}$$

are preserved.

Division with qualitative values which include 0 is not allowed. This means that the divisor $\underline{Y}^*$ in $\underline{X}^* \oslash \underline{Y}^*$ must be always strictly positive or strictly negative qualitative value, and only one element y_1 or y_3 of the fuzzy vector $\underline{Y}^* = [y_1\ y_2\ y_3]$ has to be equal to 0. This is one of the reasons why second approach to interpretation of ambiguous values is more convenient.

If the divisor $\underline{Y}^* = [y_1\ y_2\ y_3]$ is a strictly positive or negative qualitative value, the multiplication inverse $(\underline{Y}^*)^{-1}$ could be defined as:

$$(\underline{Y}^*)^{-1} = [y_{1I}\ y_{2I}\ y_{3I}] \tag{14}$$

where for $i = 1, 2, 3$

$$y_{iI} = \begin{cases} (1/y_i)/y, & \text{if } y_i \neq 0 \\ 0, & \text{if } y_i = 0 \end{cases} \tag{15}$$

and

$$y = \bigvee_i (1/y_i), \text{ for } y_i \neq 0. \tag{16}$$

For such definition of multiplication inverse the equation

$$\underline{Y}^* \otimes (\underline{Y}^*)^{-1} = [0\ 1\ 1] \tag{17}$$

is always satisfied.

Now division could be defined as

$$\underline{X}^* \oslash \underline{Y}^* = \underline{X}^* \otimes (\underline{Y}^*)^{-1} \tag{18}$$

Properties as

$$((\underline{Y}^*)^{-1})^{-1} = \underline{Y}^* \tag{19}$$

$$\underline{X}^* \oslash (\underline{Y}^* \otimes \underline{Z}^*) = (\underline{X}^* \oslash \underline{Y}^*) \otimes ([0\ 1\ 1] \oslash \underline{Z}^*) = (\underline{XC}^* \oslash \underline{Y}^*) \otimes (\underline{Z}^*)^{-1} \tag{20}$$

are preserved, but the same as for multiplication $\otimes$ the distributivity law is not valid.

The operation division satisfies a lot of common knowledge laws. One such example is that the result of division of two positive values bigger than unity must be smaller than unity if the divisor is bigger than dividend. $[0\ 0.5\ 1] \oslash [0\ 0.2\ 1] = [0\ 1\ 0.4]$ is a typical numerical illustration.

4. APPLICATION OF FUZZY QUALITATIVE ARITHMETICS

Fuzzy qualitative arithmetics proposed in this paper could be used for many qualitative reasoning tasks. Typical example is qualitative analysis of complex ecosystems, biological or sociological systems. Usually our knowledge is not adequate for construction of precise, quantitative model of such systems, but some qualitative information are usually disposable. Directed graphs could be used for visualization of interactions inside the system and fuzzy qualitative values could be used to characterize the strength of system variables interactions [9].

Proposed qualitative arithmetics based on fuzzy set theory has allowed us to develop an interactive software QUANA for qualitative system analysis [10]. Typical examples of analysis are qualitative analysis of system sensitivity to independent parameter change, qualitative analysis of pulse propagation through the system, qualitative analysis of system balance, etc. Software was successfully used in marine ecology problems. Eutrophication of semi-closed marine basins was one typical example modelled and analysed using QUANA.

5. CONCLUISION

Qualitative reasoning is very important in our everyday life, so it is not surprising that it has been intensively studied. The simplest quantity spaces that could be used in qualitative reasoning is the sign quantity space $Q = \{+, -, 0, ?\}$. One of the weak points of reasoning with this quantity space is its inherent ambiguity in connection with the addition of positive and negative values. This ambiguity could be resolved only if at least partial knowledge about variable magnitudes is known.

The paper describes how this knowledge, which is in lot of cases vague and subjective, could be mathematically represented and used in qualitative reasoning. Approach is based on the theory of fuzzy sets. The variable qualitative values are modelled with fuzzy sets and algebraic operations are defined in terms of manipulation with fuzzy sets membership functions. Obtained results are quite satisfactory and encouraging.

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