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FUZZY FEEDFORWARD - FEEDBACK COMPUTER CONTROL OF PROCESSES WITH A FACTOR OF UNCERTAINTY

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ABSTRACT

The concept of fuzzy feedforward-feedback control is presented as a new approach to the control of processes with the factor of uncertainty, particulary suitable when a high degree of process invariance to disturbances has to be achieved.

Few methods for formulation fuzzy feedforward and feedback control actions are discussed, and special attention is given to the problem of evaluation of composite, feedforward-feedback control actions.

Selected numerical example illustrates the proposed idea.

SCOPE

Two main classes of processes with the factor of uncertainty can be distinguished, depending how the uncertainty is treated. If the factor of uncertainty appears only in the expression of input and state data, while the model of the process is given as a function, we are speaking about processes with external factor of uncertainty. Otherwise, if the input and state data can be precisely described, but there is uncertainty and doubt in input-output relationship, for example due to the complexity of the process, we are speaking about processes with internal factor of uncertainty.

In both cases the application of conventional control did not give desired quality. In recent years the use of linguistic modelling and approximate reasoning, based on the fuzzy set theory has been proved as a favorable approach to control of such processes. Many papers dealing with the construction, modification and analysis of the fuzzy logic controllers and their application to the feedback control have been published [1]. We have studied the feedforward tasks and we use the methods of linguistic modelling and fuzzy control as a new basis of the feedforward-feedback control. Namely, when the control goal is to achive a high degree of process invariance to disturbances*, the feedforward control has been proved as better solution then the feedback one [2]. But the feedforward control based on the fuzzy model of the process could not ensure the highest possible degree of the process invariance to disturbances, not only because of the fuzzy model imperfection, but also because of the influence of the unobservable disturbances. To overcome this problem and to improve the degree of the process invariance, the feedback controller is also added, and composite (feedforward-feedback) control is obtained.

Distinguishing the space of disturbance input, Ω, manipulated input, U, and the space of controlled output, Y, the process can be defined as a complex:

$$M = \{\Omega,\ U,\ Y,\ R\} \qquad (1)$$

where R is a relation beetwen the sequence of controlled output, disturbance input, manipulated input and actual controlled output.

For processes with external factor of uncertainty, R is a function, and (1) represents the deterministic process, but the actual values of the sequence of controlled output, disturbance input and manipulated input are fuzzy sets defined on the spaces Y, Ω and U.

Otherwise, for the processes with internal factor of uncertainty R is a fuzzy relation, and (1) represents the nondeterministic process. The actual values of the sequence of controlled output, disturbance input and manipulated input are usualy the single elements belonging to Y, Ω and U.

There exist the cases when the process has external and internal factor of uncertainty, e.g. it is a nondeterministic, relational process and actual values of input and output variables are fuzzy sets defined on the appropriate spaces.

In this paper an approach to fuzzy feedforward-feedback computer control of such process is presented and the control procedures are developed. After the presentation of the theoretical bacground the methods have been illustrated by simulation of the control of simple industrial process.

CONCLUSION AND SIGNIFICANCE

There are many real situations in process industry where conventional control techniques did not give desired quality, mostly because of inadequate knowledge, doubt and uncertainty in the expression of input and state data and/or in the input-output relationships. However, even a crude knowledge of the values of input and state variables and/or of the process behaviour is sufficient to apply the principles of fuzzy control, which has been proved as a efficient control technique for such processes.

Since the first work of Mamdani in 1974. there are many theoretical and experimental studies and practical applications of fuzzy control, but they have been primarily concerned with various feedback control tasks.

This paper presents a method of fuzzy reasoning for feedforward-feedback computer control and shows how to derive control acitons when as a control criterion the process invariance to disturbances is considered. Using the proposed methods adequate control results can be obtained even, when only a rough and subjectively interpretated information on the values of process variables and process behaviour is attainable.

The aim of this work is not to emulate conventional feedforward-feedback deterministic or stohastic control, but to extend the area of ap-

* Under term disturbances we mean the undesirable changes of process input variables.

plication of automatic control to these situations where convetional methods fail.

FUZZY FEEDFORWARD - FEEDBACK CONTROL SYSTEM

The fuzzy feedforward-feedback control system, structured according to Fig.1.

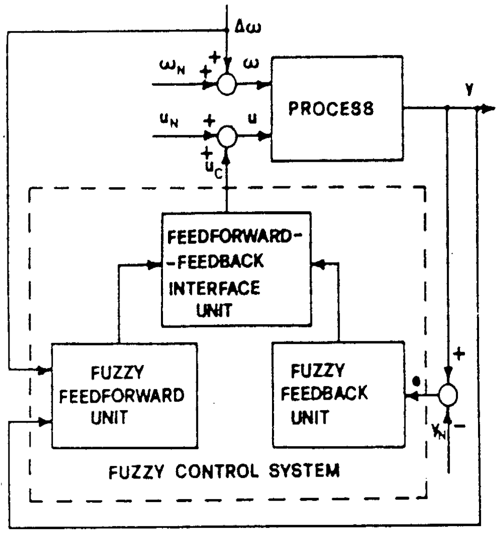

Fig.1. Fuzzy feedforward-feedback control system

is conceived as an assembly of three units:

- fuzzy feedforward unit
- fuzzy feedback unit, and
- feedforward-feedback interface unit.

The feedforward and feedback part of fuzzy control system acts independently and its internal structure depends of the feedforward-feedback interface unit. Two approaches can be distinguished:

a) feedforward-feedback interface unit perform convetional addition, and
b) feedforward-feedback interface unit perform fuzzy addition.

In the first case, both, the fuzzy feedforward and the fuzzy feedback controllers can be designed according to Fig.2.

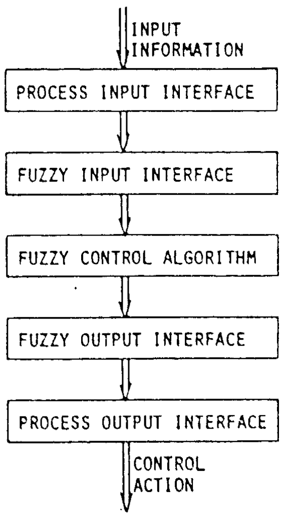

Fig.2. The internal structure of fuzzy feedforward and fuzzy feedback control unit

The units for input and output interface permit the flow of information beetwen the process and the computer which controls the process according to the fuzzy control algorithms.

The process input interface consists of measuring devices, but when the variables are scarcelly measurable, the ability of an operator to observe and express linguistically the actual values of variables may be considered.

In the fuzzy input interface these data are represented, according to fuzzy discretisation method [3] or method of discrete intervals [4], by fuzzy vectors.

The fuzzy output interface converts an output fuzzy vector into a single value which presents the control action of each controller. For example the "mean value method" can be used by means of which the output is calculated as a mean value of the elements in the output fuzzy vector.

The process output interface associates this control action with a real value, which is the adequate feedforward, u_{FF}, or feedback, u_{FD} control command, so u_{FF} and u_{FD} are input information of the feedforward-feedback interface unit.

In the second case, the feedforward and feedback part of the fuzzy control system are designed without output interfaces, and the input information of the feedforward-feedback interface unit are fuzzy sets u^*_{FF} and u^*_{FD}.

FORMULATION OF FUZZY FEEDFORWARD AND FUZZY FEEDBACK CONTROL ACTIONS

Two methods for formulation of fuzzy feedforward control actions can be distinguished. In the first method the fuzzy control action, u^*_{FF}, is calculated from the fuzzy model of the process.

The fuzzy model of the process suitable for application in feedforward control can be defined as a complex [5]:

$$M = \{\Omega, U, Y, R^*\} \qquad (2)$$

where Ω, U and Y are real spacies defined in (1) and R^* is a fuzzy relation which describes the relationship beetwen process input and output variables.

If we concentrate ourselves, for the sake of simplicity, on the first order systems, supposing that process input and output variables have nominal operating values (Fig.1), and that Ω, U and Y are discrete real spaces, the fuzzy relation is a discrete fuzzy relations $\underline{R}^*(\Delta\omega, \Delta u, y, \Delta y)$, and the process can be described with fuzzy difference relational equation:

$$\underline{\Delta y}^*_n = \underline{\Delta\omega}^*_n \circ \underline{\Delta u}^*_n \circ \underline{y}^*_n \circ \underline{R}^* \qquad (3)$$

where o is max-min composition operator.

The fuzzy feedforward control action in each instant of time nT, $\underline{u}^*_{FFn}$, is obtained as a solution of the equation (3), $\underline{u}^*_{FFn} = \underline{\Delta u}^*_n$, where $\underline{\Delta y}^*_n$ corresponds to the real value 0, or linguistic value "zero".

Various methods can be used for the exact or approximate solution of the equation (3) [6].

For the processes with internal factor of uncertainty, $\underline{\Delta y}^*_n$, $\underline{\Delta\omega}^*_n$ and $\underline{y}^*_n$ will be degenerated fuzzy sets, and R^* will be the fuzzy relation, and vice versa for the processes with exteral factor of uncertainty, $\underline{\Delta y}^*_n$, $\underline{\Delta\omega}^*_n$ and $\underline{y}^*_n$ will be real fuzzy sets, and $\underline{R}^*$ wil be the Boolean relation.

According to the second method, the fuzzy control action, $\underline{u}^*_{FF_n}$ is calculated from the equation:

$$\underline{u}^*_{FF_n} = \underline{\Delta\omega}^*_n \circ \underline{y}^*_n \circ \underline{G}^*_{FF} \tag{4}$$

where $\underline{G}^*_{FF}$ is the fuzzy gain discrete relation of the fuzzy feedforward control for a given goal (the achivement of process invariance to disturbances). $\underline{G}_{FF}$ is the fuzzy model of a feedforward controller fulfilling the same role as a controller in a multivariate linear system [7].

The fuzzy relation $\underline{G}^*_{FF}$ is calculated off line as a solution of the set of equations:

$$\underline{\Delta y}^* = (\underline{\Delta\omega}^*)_i \circ [(\underline{\Delta\omega}^*)_i \circ (\underline{y}^*)_i \circ \underline{G}^*_{FF}] \circ (\underline{y}^*)_i \circ \underline{R}^*, \quad i=1,\dots,I \tag{5}$$

where $\underline{\Delta y}^*$ corresponds to the real value 0, or linguistic value "zero", and $\{(\underline{\Delta\omega}^*)_i\}$ and $\{(\underline{y}^*)_i\}$, $i=1,\dots,I$ are *a priori* defined adequate sets of fuzzy vectors.

For example the set $\{(\underline{\Delta\omega}^*)_i\}$ could correspond to the set of linguistic values {NB,NM,NS,ZE,PS,PM,PB} where NB means "negative big", NM-"negative medium", NS-"negative small", ZE-"zero", and the the same for positive values.

Pedrycz approximative method [8] can be used for the calculation of the relation $\underline{G}^*_{FF}$.

In both cases we need fuzzy relation $\underline{R}^*$ which describes the relationship between process input and output variables. R^* can be obtained by various identification methods [5] using either formal or linguistic approach. Input information to the identification procedure may be real (but imprecise) date, or linguistic description of input-output relationships in the form of causal statements, as for example:

"If in the moment nT, $\Delta\omega$ is "negative small", Δu is "positive big" and y is "very big", then the output increment Δy will be "positive small".

The fuzzy feedback control action, $\underline{u}^*_{FD_n}$, is calculated by the equation:

$$\underline{u}^*_{FD_n} = \underline{e}^*_n \circ \underline{\Delta e}^*_n \circ \underline{G}^*_{FD} \tag{6}$$

where $\underline{e}^*$ and $\underline{\Delta e}^*_n$ are fuzzy vectors of the controlled output error and its increment, obtained either by the operator's observation or by calculation and fuzzification from the actual values of the controlled output, its previous and desidered values. $\underline{G}^*_{FD}$ is the fuzzy matrix of the fuzzy feedback controller obtained by any one of the existing synthesis methods for the fuzzy feedback controllers (for example [9,10]).

FEEDFORWARD-FEEDBACK INTERFACE UNIT

Feedforward-feedback interface unit may be designed as a unit with:

a) conventional addition, or
b) fuzzy addition.

In the first case, the input information to the interface unit are yet interpretated real values of feedforward and feedback control commands u_{FF_n} and u_{FD_n}, so the only purpose of the unit is conventional addition of these values

$$u_{C_n} = u_{FF_n} + u_{FD_n} \tag{7}$$

In the second case, input information to the feedforward-feedback interface unit are fuzzy sets $u^*_{FF_n}$ and $u^*_{FD_n}$. The purpose of this unit is now the calculation of the fuzzy set, $u^*_{C_n}$

$$u^*_{C_n} = u^*_{FF_n} \oplus u^*_{FD_n} \tag{8}$$

its interpretation and association with a real value, u_C, which is adequate command to the proces. $\oplus$ is the operator of the extended addition.

In this paper we shall give more attention to this second case.

The Zadeh extension principle [11] allows us to calculate the sum of two fuzzy sets. In practical application the finite and discrete supports of fuzzy sets are considered, so fuzzy sets may be expressed with fuzzy vectors

$$\underline{u}^*_{FF_n} = [u_{FF_i}], \quad i = 1,\dots,I$$

$$\underline{u}^*_{FD_n} = [u_{FD_j}], \quad j = 1,\dots,J$$

$$\underline{u}^*_{C_n} = [u_{C_k}], \quad k = 1,\dots,K$$

The calculation procedure depends about the kind of interaction between fuzzy vectors $\underline{u}^*_{FF_n}$ and $\underline{u}^*_{FD}$. In our case, we may suppose that fuzzy vectors $\underline{u}^*_{FF_n}$ and $\underline{u}^*_{FD_n}$ will be noninteractive, because feedforward and feedback units act independently, so $\underline{u}^*_{C_n}$ can be calculated with equation

$$u_{C_k} = \sup_i \min\,(u_{FF_i}, u_{FD_{(k-i)}}), \; k = 1,\dots,K \tag{9}$$

Dubois and Prade [11] proposed a general algorithm for the computation of extended addition, but because of its redundancy, it is not applicable for our purpose. We have developed more efficient and nonredundant algorithm based on the equation (9), and its principles are given in Appendix.

The final result of extended addition is the fuzzy vector $\underline{u}^*_{C_n}$. After the interpretation procedure, to the single value which presents the fuzzy vector $\underline{u}^*_{C_n}$, adequate real value has to be associated, and that is the control command, u_{C_n}, to the process.

NUMERICAL EXAMPLE

Fig.3. illustrates a hot-water heater that mixes cold water with steam to produce hot water. The hot water temperature is controlled output, steam flow manipulated input and cold water flow disturbance input. The deterministic representation of considered process is also given, because the process was simulated on digital computer.

Let us suppose that nominal operating values of process variables are $F_N=10$, $Q_N=T_N=5$, and that instead of deterministic model only the set of cause-effect numeric data are known:

$S_D=\{(\Delta Q,\Delta T)\}=\{(-4.5,0.18),(-2.1,0.051),(-0.9,0.015),(0,0),(1.3,-0.05),(3,-0.14),(5,-.21)\}|F=F_N$

$S_M=\{(\Delta F,\Delta T)\}=\{(-4.5,-0.42),(-2,-0.22),(-1,-0.15)(0,0),(0.5,0.1),(3,0.28),(4.7,0.43)\}|T=T_N.$

Each pair can be linguisticly interpretated with conditional statement, for example statement for pair $(\Delta Q,\Delta T)=(3,-0.14)$ is:

"If steam flow is on nominal value and cold water flow increase 3 units above nominal value, then the hot water temperature will decrease 0.14 units 0.5 minutes later".

The process behaviour can be described with two simple fuzzy diference equations $\underline{\Delta T}^*=\underline{\Delta Q}^* \circ \underline{R}^*_D$ and $\underline{\Delta T}^*=\underline{\Delta F}^* \circ \underline{R}^*_M$, because in process description the principle of superposition can be applied.

Defining appropriate basic fuzzy sets for fuzzy

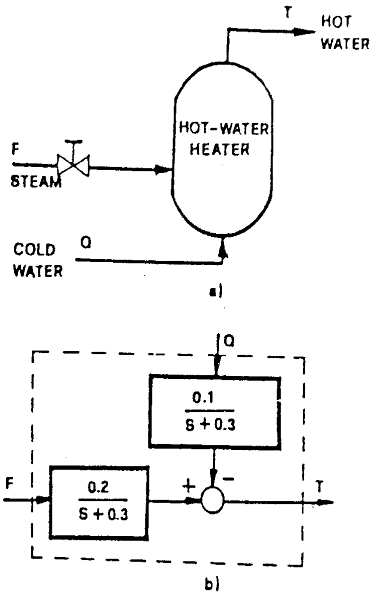

Fig.3.a) The hot-water heater
b) The deterministic represent of the considered process (time constants in minutes)

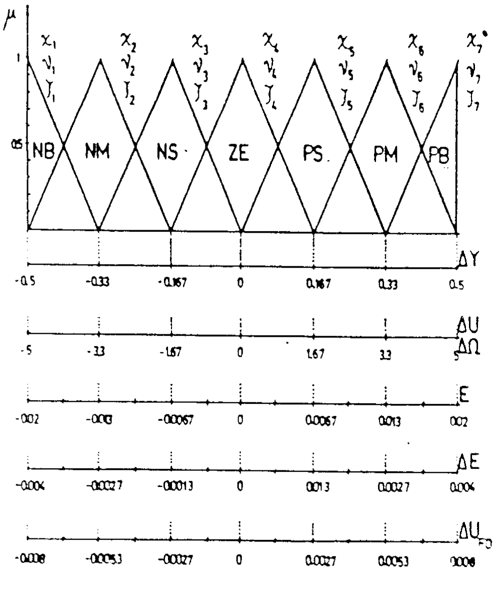

Fig.4. Input-output data, basic fuzzy sets of fuzzy discretisation with its linguistic labels and definition of fuzzy sets for variables e, Δe and Δu_{FD} of the fuzzy feedback controller (NB - negative big, NM-negative medium, NS-negative small, ZE-zero, PS-positive small, PM-positive medium, PB-positive big)

discretisation by Fig.4. and using formal approach in identification procedure, from the sets S_D and S_M fuzzy matrices $\underset{\sim}{R}^*_D$ and $\underset{\sim}{R}^*_M$ can be calculated [5].

The fuzzy feedforward control action $\underset{\sim}{u}^*_{FFn}$ is obtained as a solution of the equation

$$\underset{\sim}{\Delta Q}^*_n \circ \underset{\sim}{R}^*_D = \underset{\sim}{u}^*_{FFn} \circ \underset{\sim}{R}^*_M \tag{10}$$

$$\underset{\sim}{R}^*_D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.204 \\ 0 & 0 & 0 & 0 & 0.3 & 1 & 0.796 \\ 0 & 0 & 0.695 & 1 & 0.162 & 0.162 & 0 \\ 1 & 1 & 0.09 & 0 & 0 & 0 & 0 \\ 0.078 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{R}^*_M = \begin{bmatrix} 0.52 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.48 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.68 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.68 & 0.42 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.58 \end{bmatrix}$$

For fuzzy feedback control the Andersen-Nielsen method [12] in non adaptive form is used. Table 1. gives the set of linguistic rules of the applied fuzzy feedback controller.

Δe^L \ e^L	PB	PM	PS	ZE	NS	NM	NB
PB	NB	NM	NM	NS	ZE	ZE	ZE
PM	NB	NM	NB	NM	ZE	ZE	PS
PS	NB	NB	NS	ZE	PS	PS	PM
ZE	NB	NM	NS	ZE	PS	PM	PB
NS	NM	NS	NS	ZE	PS	PB	PB
NM	NS	ZE	ZE	PM	PB	PM	PB
NB	ZE	ZE	ZE	PS	PM	PM	PB

Table 1. Δu^L_{FD}

The real spaces E, ΔE and ΔU_{FD} are represented by 13 discrete points and the appropriate fuzzy sets are given on Fig.4.

The output of the fuzzy feedback controller is the control action increment Δu_{FDn} or $\underset{\sim}{\Delta u}^*_{FDn}$, so cumulative action have to be calculated. For conventional addition the fuzzy feedforward-feedback control action is obtained by the equation

$$u_{C_n} = u_{FF_n} + u_{FD_{n-1}} + \Delta u_{FD_n} \tag{11}$$

and for fuzzy addition by the equation

$$\underset{\sim}{u}^*_{C_n} = \underset{\sim}{u}^*_{FF_n} \oplus (\underset{\sim}{u}^*_{FD_{n-1}} \oplus \underset{\sim}{\Delta u}^*_{FD_n}) \tag{12}$$

Fig.5. shows the response of the process under the control of fuzzy feedforward-feedback controller with conventional and fuzzy addition in interface unit. The disturbance input was $\Delta Q = -5 \cdot \sin 0.002\, t$. For the finite time interval of 100 minutes (sampling interval 0.5 minutes) the integral of squared errors for conventional addition was 0.5 and for fuzzy addition 0.8.

The results are comparable, but the conventional addition has a little advantage.

In order to compare the fuzzy control approach with conventional control approach a linear regression model for the same input-output data (sets S_D and S_M) is considered in the feedforward part, and Dahlin DDC algorithm based on the exact model of the process in the feedback part of the conventional controller.

Fig.6. shows the process response for fuzzy and conventional control.

For the same finite time interval, the integral of squared errors was for fuzzy control 0.5 and for conventional control 4.2. It is evident that even in this case the fuzzy approach gives better results then the conventional approach.

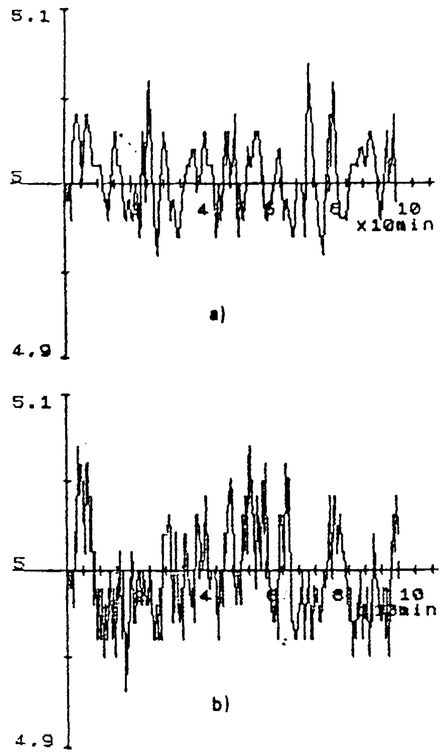

Fig.5. Process response for conventional (a) and fuzzy (b) addition in feedforward-feedback interface unit for sine disturbance

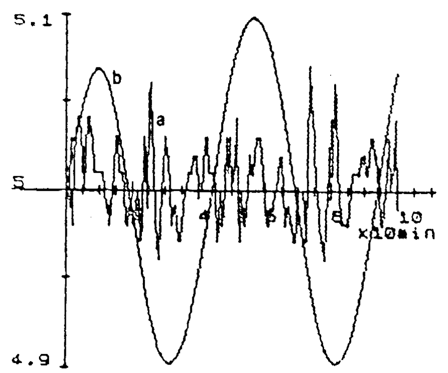

Fig.6. Process response when fuzzy (a) and conventional (b) feedforward-feedback control is applied

At the end it is important to emphasize that the real worth of the fuzzy control is in those situations when the exact mathematical model is not known and when it is impossible to define any kind of deterministic model.

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APPENDIX

Algorithm for additon of noninteractive fuzzy numbers with finite and discrete supports

Let

$$\underline{A}^* = [a_i] \ , \ i = 1,\dots,I$$

and

$$\underline{B}^* = [b_j] \ , \ j = 1,\dots,J$$

be two noninteractive fuzzy numbers with finite and discrete supports A and B. The Zadeh extension principle allows us to calculate the sum of these fuzzy numbers

$$\underline{C}^* = \underline{A}^* \oplus \underline{B}^* = [c_k] \ , \ k = 1,\dots,K$$

where

$$c_k = \sup_i \min (a_i, \ b_{i-k})$$

The consistency of the representation of the real values has to be preserved, so the support of the fuzzy vector $\underline{C}^*$ and the number of its elements, K, may not be arbitrary chosen.

Let us suppose that the supports of the fuzzy vector $\underline{A}^*$ and $\underline{B}^*$ are sets of integers $A = \{a_m,\dots,a_M\}$ and $B = \{b_m,\dots,b_M\}$. The support of the fuzzy vector $\underline{C}^*$ has to be the set of integers $C = \{c_m,\dots,c_M\}$, too, where $c_m = a_m + b_m$ and $c_M = a_M + b_M$. For example if $A = B = \{-n,\dots,n\}$, then $C = \{-2n,\dots,2n\}$, or if $A = \{1,\dots,7\}$ and $B = \{-3,\dots,3\}$ then $C = \{-2,\dots,10\}$.

The number of elements of the fuzzy vector $\underline{C}^*$ has to be $K = I + J - 1$.

The algorithm for addition of noninteractive fuzzy numbers with finite, discrete supports consists of two steps:

a) calculation of fuzzy matrix $\underline{W}^*=[w_{ji}]$ where

$$w_{ji} = \min\,(a_i,b_j),\quad i=1,\dots,I,\ j=1,\dots,J$$

b) calculation of elements of the resulting fuzzy vector $\underline{C}^*=[c_k]$:

$$c_k = \max_m w_{m,(k-m+1)},\quad k=1,\dots,K$$

where m is the index, and it takes values on the finite index set M. Cardinality of M, for appropriate k, is equal to the number of elements on the reverse diagonals of the matrix $\underline{W}^*$.

Three situations may occur:

a) $I = J$
$K = 2I - 1 = 2J - 1$

$m = 1,\dots,k$, for $1 \le k \le I$
$m = k-I+1,\dots,I$, for $I \le k \le K$

b) $I < J$
$K = I + J - 1$

$m = 1,\dots,k$, for $1 \le k \le I$
$m = 1,\dots,I$, , for $I \le k \le J$
$m = k-J+1,\dots,I$, for $J \le k \le K$

c) $I > J$
$K = I + J - 1$

$m = 1,\dots,k$, for $1 \le k \le J$
$m = k-J+I,\dots,k$, for $J \le k \le I$
$m = k-J+1,\dots,I$, for $I \le k \le K$

Example:

$I = J = 5$

$K = 2I - 1 = 9$

$A^* = [0\ \ 0\ \ 0.5\ \ 1\ \ 0.5]$

$B^* = [0\ \ 0\ \ 0\ \ 0.5\ \ 1]$

$$\underline{W}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1 & 0.5 \end{bmatrix}$$

$\underline{C}^* = \underline{A}^* \oplus \underline{B}^* = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.5\ \ 0.5\ \ 1\ \ 0.5]$

For example for $k=7$, $M=\{3,4,5\}$, and

$c_7 = \max\ (w_{35}, w_{44}, w_{53}) = \max(0, 0.5, 0.5) = 0.5$

If $A = B = \{-2,-1,0,1,2\}$ it follows that $C=\{-4,\dots,0,\dots,4\}$. $\underline{A}^*$ corresponds to the fuzzy number 1^*, $\underline{B}^*$ to the fuzzy number 2^*, and $\underline{C}^*$ to the fuzzy number $3^* = 1^* \oplus 2^*$.

N.B. The similar algorithm can be used for the subtraction of fuzzy numbers $\underline{C}^*=\underline{A}^* \ominus \underline{B}^*$ with following exceptions:

a) The bounds of the support C of $\underline{C}^*$ are

$$c_m = a_m - b_M \text{ and } c_M = a_M - b_m,\quad \text{and}$$

b) $c_k = \max\limits_m\ w_{(J-k+m),m},\quad k=1,\dots,K$

m is the same, so this mean that for each k, we take the maximum value of the elements on the main diagonals.