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QUALITATIVE REASONING AND FUZZY SET THEORY

D.STIPANIČEV*

QUEEN MARY COLLEGE
UNIVERSITY OF LONDON
Mile End Road, LONDON E1 4NS

J.EFSTATHIOU

THE OXFORD TRUST
STEP CENTRE
Osney Mead, OXFORD OX2 0ES

ABSTRACT. To resolve the inherent ambiguity of qualitative reasoning with sign values an approach based on the fuzzy set theory is proposed. It can be used when at least vague and subjective knowledge about variables magnitudes or their relative magnitudes is known. Fuzzy sets are used to model these qualitative values and algebraic operations are defined in terms of manipulations with their membership functions. The results of reasoning using this approach coincide quite satisfactory with our common sense.

KEY WORDS. qualitative reasoning, fuzzy sets, magnitude reasoning, order of magnitude reasoning

1. INTRODUCTION

In reasoning about physical systems, humans prefer to use qualitative values instead of quantitative. Our ability to create qualitative models of the world and to use them in reasoning about the world is maybe one of the reasons why humans cope (relatively) successfully with real world problems and situations. Because of that, it is not surprising that many attempts have been made to understand the qualitative way of thinking and reasoning and to create an AI system with the same capabilities. That is specially the case in the last decade when a number of different approaches known as "qualitative reasoning" have been proposed [1].

Qualitative reasoning with all its advantages, particularly in dealing with the plant or organisational complexity also has a few weak points. One of them is its inherent ambiguity if the elementary quantity space with sign values positive, negative and zero is used. In lot of qualitative inferences with these values the obtained result is "?" what means that the exact answer could not be find. To solve this problem a number of approaches have been proposed, from subdividing the positive and negative line into more precise subintervals by introducing new landmarks besides the existing landmarks $-\infty$,0 and ∞ [1,2], till the definition of reasoning with order of magnitudes [3,4,5]. But then some other problems occur, as for example the problem of finiteness of quantity space or lack of addition associativity.

* Currently a British Council academic visitor at Queen Mary College. Permanent address Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, University of Split, R.Boškovića bb, 58000 Split, Yugoslavia

214

Here an alternative approach to this problem will be proposed based on fuzzy set theory. It is important to stress that the reasons behind the introduction of the fuzzy set theory were similar to reasons behind the introduction of qualitative reasoning. In both cases qualitative instead of quantitative was the prime motivation, although the fuzzy set theory was introduced to deal with process complexity, with ill-understood phenomena and uncertainty, and the qualitative reasoning was introduced to deal with plant or organisational complexity.

The task of this paper is not to give answers to all open questions, but to show how the ideas from the fuzzy set theory could be used to define and solve some qualitative reasoning problems. Although both approaches are qualitative in nature, it seems that the qualitative community is strictly divided into two streams and that neither of them use results already obtained on the other side.

2. QUALITATIVE REASONING AND FUZZY ARITHMETIC

One of the most simple quantity spaces which could be used in qualitative reasoning is a sign quantity space with three values : positive (+), negative (-) and zero (0). In 1964, *Harrary et al.*[6] introduced the sign arithmetic and defined addition ⊕† and multiplication ⊗† of sign values with Cayley tables (1) and (2) using common sense about calculations with signs. Instead of three values quantity space they used a four value quantity space Q = {+,-,0,?}, where ? means 'ambivalent', 'ambiguous', 'indeterminate' or 'any sign value'. Without ? it was not possible to define addition as a closed arithmetic operation.

⊕	0	+	-	?
0	0	+	-	?
+	+	+	?	?
-	-	?	-	?
?	?	?	?	?

(1)

⊗	0	+	-	?
0	0	0	0	0
+	0	+	-	?
-	0	-	+	?
?	0	?	?	?

(2)

More recently *Struss* [2] introduced in qualitative reasoning the interval arithmetic as a more general approach than the sign arithmetic, because it allows to use more complicated quantity space.The core of his approach is representation of qualitative values with intervals, so the quantity space is the set of open and closed intervals:

$$Q_B = \{(a,b) \mid a,b \in B,\ a<b\} \cup \{[a,a] \mid a \in B/\{-\infty,\infty\}\} \qquad (3)$$

For sign quantity space Q = {+,-,0,?} the set of landmarks is B = {-∞,0,∞}, so Q_B = ={(0,∞),(-∞,0),[0,0],(-∞,∞)}. Addition ⊕ and multiplication ⊗ are defined in terms of well known interval manipulations.

The algebraic structure (Q,⊕,⊗) has interesting mathematical properties, because it belongs to the path algebras [7]. ⊕ is an associative, commutative and idempotent operation. ⊗ is an associative and commutative operation and it is distributive over ⊕, and both addition and multiplicative identities and zeros exist.

If the landmarks are introduced the situation is not any more so nice, and a lot of problems arise [2], as for example the problem of finiteness of the quantity space, the problem of definition of operation subtraction etc. Some of these problems could be resolved by modifying interval arithmetics by restricting the quantity space to a set of elementary intervals on Q_B, but then another problems arise, typically the operation addition is not any more associative.

For many qualitative reasoning applications the elementary quantity space Q = {+,-,0,?} is precise enough and even sometimes the most complicated quantity space that could be used, the cases exist when our knowledge is more complete, so the more complicated quantity space could be beneficial. Its main advantage is that it can greatly reduce the problem of ambiguity.

† Here for addition and multiplication the notations ⊕ and ⊗ from the qualitative reasoning theory will be used. It is important not to confused them with the bounded sum and bounded product from the fuzzy set theory.

To avoid problems created by introducing new landmarks a different approach will be suggested based on the idea of fuzzy sets. The core of this approach is to use the fuzzy sets to model the qualitative values and to define algebraic operations in terms of manipulations with their membership functions.

2.1. Definition of qualitative values with fuzzy sets

Instead of introducing new landmarks to expand the quantity space and define more qualitative values, these can be defined using the same landmarks as in the sign quantity space B = {-∞,0,∞} and introducing the fuzzy sets over them. This means that the qualitative values are defined with the discrete fuzzy sets whose support set is a three element set B. The value of the membership function could be seen as a plausibility degree to which the qualitative value is more or less close to the appropriate landmark. For example the membership function of the big negative qualitative value could be expressed using the usual notation from the fuzzy set theory as $1/-\infty + 0.2/0 + 0/\infty$, or more conveniently in the form of fuzzy vector [1 0.2 0] where the first element correspond to the support element -∞, the second element to 0 and the third to ∞. Similarly a smaller qualitative value, but still negative, could be expressed as [1 0.5 0], and approximately the same positive qualitative value as [0 0.5 1]. Note that [0 0.5 1] represents the qualitative value smaller than [0 0.2 1].

One restriction to the representation of qualitative values with fuzzy sets is introduced. That is that the fuzzy set must be normalised, so at least one of the elements of the fuzzy vector must be equal to 1.The reason for this is because than the interpretation of the results is easier, as will be shown later.

The sign values from the sign quantity space Q = {+,-,0,?} are defined simply using only boundary values 1 or 0 for values of the membership function.The fuzzy set representation of '+' is [0 1 1], the representation of '-' [1 1 0], the representation of '0' [0 1 0] and the representation of '?' [1 1 1] . This can be explained that for simple sign quantity space we don't know how close is the variable to the landmark. We only know that '-' is somewhere between landmarks -∞ and 0, so to both of them we assign value 1. For '+' the situation is similar but the value '0' which correspond to the closed interval [0,0] is modeled with [0 1 0], and that means that the value coincide with the landmark 0. That is the reason why '?' is not modeled with [1 1 1], because we want to emphasize that '?' is a union of all three sign values.

As in all other cases where fuzzy sets are used to model the real world, the question arises how to assign membership values. The easiest way is to do that subjectively as a result of human observations. Qualitative reasoning is usually connected with changes of variables and humans can easily judge (if the time constants are acceptable to human perception) are the changes big, very big, small or maybe more or less small. Now a theory of linguistic modelling based on fuzzy set theory could be used, and only the membership functions of primary linguistic terms defined in advance, as for example "positive big" with [0 0.2 1] or "negative medium" with [1 0.5 0]. For hedges standard operations from linguistic modelling theory could be used. Typical example is to use 2nd power for "very", so "very positive big" could be represented with [0 0.04 1], if "positive big" is [0 0.2 1].

This was the situation when reasoning is performed with the absolute magnitudes, but the real power of proposed approach is reasoning with relative magnitudes or reasoning with order of magnitudes. In that case it is even more easy to define membership functions. That will be explored more deeply in the Ch.2.3. and Ch.3. Here first the algebraic operations with the fuzzy vectors which represent qualitative values will be introduced.

2.2 Arithmetic with fuzzy qualitative values

Arithmetic operations must be defined such that for boundary case, when the quantity space is restricted to Q = {+,-,0,?} the results coincide with the results obtained using definitions from the qualitative reasoning theory: for addition and multiplication tables (1) and (2) and for subtraction and division formulas

$$x \ominus y = x \oplus (-\otimes y), \quad x,y \in Q \qquad (4)$$

$$x \oslash y = x \otimes y, \quad x \in Q, \quad y \in \{+,-\} \qquad (5)$$

Division with 0 is of course not allowed, but as ? incorporate the value 0 division with ? is also not defined.

Addition and multiplication

Let us start with the operation addition. According to Zadeh's extension principle the sum of two non-interacting fuzzy sets with membership functions $X^*(x)$ and $Y^*(y)$ is a fuzzy set U^* with membership function $U^*(u)$:

$$U^*(u) = \bigvee_{x+y=u} (X^*(x) \wedge Y^*(y)) \quad (6)$$

where V is max and Λ min. In our case all fuzzy sets including the resulting fuzzy set have the same support set $\{-\infty, 0, \infty\}$. According to our knowledge about extended algebra of limits the following assumptions hold:

a) $(-\infty)+(-\infty)$ and $(-\infty)+0$ give $-\infty$,

b) $(\infty)+(\infty)$ and $(\infty)+0$ give ∞, and

c) 0 can be obtained only as a result of 0+0, because $(-\infty)+(\infty)$ is indeterminate case,

so the addition ⊕ of two fuzzy vectors $\underline{X}^* = [x_1\ \ x_2\ \ x_3]$ and $\underline{Y}^* = [y_1\ \ y_2\ \ y_3]$ could be defined, using the extension principle, as

$$\underline{U}^* = \underline{X}^* \oplus \underline{Y}^* = [u_1\ \ u_2\ \ u_3] \quad (7)$$

where $u_1 = u_1'/u$, $u_2 = u_2'/u$, $u_3 = u_3'/u$ and

$$u_1' = (x_1 \wedge y_1) \vee (x_1 \wedge y_2) \vee (x_2 \wedge y_1) \quad (8)$$

$$u_2' = x_2 \wedge y_2 \quad (9)$$

$$u_3' = (x_2 \wedge y_3) \vee (x_3 \wedge y_2) \vee (x_3 \wedge y_3) \quad (10)$$

$$u = u_1' \vee u_2' \vee u_3' \quad (11)$$

The resulting fuzzy vector is normalised because that was the restriction introduced at the beginning.

The addition defined on such a way is a commutative and associative operation.[0 1 0] is the addition identity (neutral element), because

$$\forall \underline{X}^*\ .\ \underline{X}^* \oplus [0\ 1\ 0] = \underline{X}^* \quad (12)$$

Multiplication ⊗ could be defined using the same extension principle and knowledge that $\infty \cdot \infty$ and $(-\infty)\cdot(-\infty)$ give ∞, $(-\infty)\cdot\infty$ gives $-\infty$, $0\cdot 0$ gives 0 and $(-\infty)\cdot 0$ and $\infty \cdot 0$ are indeterminate cases, but we will propose a slightly modified definition replacing the min operator with ordinary product, so

$$\underline{V}^* = \underline{X}^* \otimes \underline{Y}^* = [v_1\ \ v_2\ \ v_3] \quad (13)$$

where $v_1 = v_1'/v$, $v_2' = v_2/v$, $v_3 = v_3'/v$ and

$$v_1' = (x_1 \cdot y_3) \vee (x_3 \cdot y_1) \quad (14)$$

$$v_2' = x_2 \cdot y_2 \quad (15)$$

$$v_3 = (x_1 \cdot y_1) \vee (x_3 \cdot y_3) \quad (16)$$

$$v = v_1' \vee v_2' \vee v_3' \quad (17)$$

The reason for this modification is because using these equations the results correspond better to our common knowledge about multiplying quantities. Let us explain that more deeply. If we multiply two big positive quantities the result usually has bigger order of magnitude than the biggest of quantities used in multiplication. For addition that is not the case. The sum normally have the same order of magnitude as the biggest quantity. If the operation min is used to define ⊗ this property is not emphasized, as it is for the equations (14)-(16). Typical example is sum and product of qualitative values [0 0.2 1] and [0 0.5 1]. The sum is equal to the bigger of them

$$[0\ 0.2\ 1] \oplus [0\ 0.5\ 1] = [0\ 0.2\ 1]$$

The same result is obtained for product using min operator, so previously explain property is not notable. But if (14)-(16) are used the product is

$$[0\ 0.2\ 1] \otimes [0\ 0.5\ 1] = [0\ 0.1\ 1]$$

and [0 0.1 1] is bigger qualitative value than [0 0.2 1] which is the biggest value used in multiplication.

For boundary case and sign quantity space our definitions give correct results and coincide with results from the Cayley table (1) and (2).

The multiplication is also commutative and associative operation, but unfortunately it is not distributive over addition, except for sign quantity space. Both multiplicative zero [0 1 0] and multiplicative identity [0 1 1] exist, because

$$\forall \underline{X}^*\ .\ \underline{X}^* \otimes [0\ 1\ 0] = [0\ 1\ 0] \quad (18)$$

$$\underline{X}^* \otimes [0\ 1\ 1] = \underline{X}^* \quad (19)$$

Multiplicative zero [0 1 0] is the same as the addition identity and it correspond to the real line element zero (0), which is also identity element in ordinary addition and zero element in ordinary multiplication on the real line. Because of this correspondence it will be interesting to analyse the similarities between the multiplicative identity [0 1 1] and its corresponding real line element, the unity (1), which has the same properties according to ordinary real line multiplication.

The fuzzy sets which represent qualitative values must be normalised. This means that for example strictly positive values could be represented on two ways with fuzzy vectors [0 x 1] and [0 1 x] where $x \leq 1$. The fuzzy vector [0 1 1] is a boundary case for both representations. As it corresponds to the number 1, it seems natural to use [0 x 1] for representing qualitative values bigger than 1 and to use [0 1 x] for representing of qualitative values smaller than 1.

Let us test this approach by introducing some common knowledge laws about multiplication:

a) The product of two positive values bigger than 1 is bigger than the biggest of them.

$$[0\ 0.2\ 1] \otimes [0\ 0.5\ 1] = [0\ 0.1\ 1]$$

b) The product of two positive values smaller than 1 is smaller than the smallest of them.

$$[0\ 1\ 0.4] \otimes [0\ 1\ 0.2] = [0\ 1\ 0.08]$$

c) The product of positive value bigger than 1 and a positive value smaller than 1 must be somewhere between them.

$$[0\ 0.2\ 1] \otimes [0\ 1\ 0.5] = [0\ 0.4\ 1]$$

Obtained results completely correspond to our common knowledge. The situation is similar for the negative qualitative values or qualitative values with mixed signs. The negative unity (-1) is [1 1 0], so the negative qualitative values with magnitudes bigger than 1 are modeled with [1 x 0] and with magnitudes smaller than 1 with [x 1 0] where $x \leq 1$.

Here is important to emphasise that it is not possible to mix simple sign values and more precise qualitative values if both are modelled with fuzzy sets. This means that in the same reasoning procedure it is not possible to use for example [0 1 1] to represent sign value '+' and [0 0.2 1] to represent 'positive big'. In calculation procedure [0 1 1] will be interpreted as the positive unity and not as the sign value '+'. All qualitative values must belong either to sign quantity space or to more precise qualitative quantity space.

Let us now return back to addition and analyse the situation which is of our prime interest: addition of positive and negative qualitative values, because one of the reasons why the whole procedure is introduced is to help in resolving the ambiguity of the original sign algebra.

Ambiguity and fuzzy qualitative values

Few examples of addition of positive and negative qualitative values are

a) [0 0.2 1] ⊕ [1 0.2 0] = [1 1 1]

b) [0 0.2 1] ⊕ [1 0.4 0] = [0.5 0.5 1]

c) [0 0.2 1] ⊕ [1 0.8 0] = [0.25 0.25 1]

d) [0 0.2 1] $\oplus$ [0.4 1 0] = [0.2 0.2 1]

e) [0 0.2 1] $\oplus$ [0.1 1 0] = [0.1 0.2 1]

f) [0 1 0.5] $\oplus$ [0.2 1 0] = [0.2 1 0.5]

The resulting fuzzy vectors have all three elements different from 0. That means a certain amount of ambiguity, but with adequate interpretation, a nonambiguous conclusion could be obtained. Typical examples of results that we want to obtain are that

- the result of addition of big positive qualitative value and the negative qualitative value of the similar magnitude must be close to zero (case a), or

- the result of addition of the big positive qualitative value and the smaller negative qualitative value is a positive qualitative value, but not so big as the first one (case b), or

- if the negative qualitative value is smaller than previously than the result must be even bigger than before (cases c,d,e).

The first idea is to solve this problem using the real number interpretation of the resulting fuzzy sets. The center of gravity method could be one of the approaches, so the resulting fuzzy vector $\underline{U}^* = [u_1\ u_2\ u_3]$ could be represent with value $\hat{u}$ calculated from the equation

$$\hat{u} = (u_1 \cdot 1 + u_2 \cdot 2 + u_3 \cdot 3)/(u_1 + u_2 + u_3) \qquad (20)$$

For $\hat{u} < 2$ the result belong to negative values and for $\hat{u} > 2$ to positive values . $\hat{u} = 2$ means that the result is exactly zero.The $\hat{u}$ values calculated for previous examples are given in Table 1 and they correspond satisfactory with our knowledge.

Another approach which is more complicated, but more interesting. It consists of transforming the fuzzy sets with all three elements different from 0 into strictly positive or strictly negative qualitative values. This means that the fuzzy set $\underline{U}^* = [u_1\ u_2\ u_3]$ where $u_1, u_2, u_3 \neq 0$ has to be transformed into fuzzy set $\underline{W}^* = [w_1\ w_2\ w_3]$ where one of the elements w_1 or w_3 is equal to 0.

Transformation formula depends of the form of the fuzzy set $\underline{U}^*$. If u_1 or u_2 are equal to 1 equations

$$w_1 = \begin{cases} 1, & \text{if } u_1 = 1 \\ 0, & \text{if } u_3 = 1 \text{ or } u_1 = u_3 = 1 \end{cases} \qquad (21)$$

$$w_2 = u_2 \qquad (22)$$

$$w_3 = \begin{cases} 0, & \text{if } u_1 = 1 \text{ or } u_1 = u_3 = 1 \\ 1, & \text{if } u_3 = 1 \end{cases} \qquad (23)$$

can be used, and if only $u_2 = 1$ the equations

$$w_1 = \begin{cases} (u_1 + u_3)/2, & \text{if } u_1 > u_3 \\ 0, & \text{if } u_1 < u_3 \end{cases} \qquad (24)$$

$$w_2 = 1 \qquad (25)$$

$$w_3 = \begin{cases} 0, & \text{if } u_1 > u_3 \\ (u_1 + u_3)/2, & \text{if } u_1 < u_3 \end{cases} \qquad (26)$$

For previous examples results of transformation is shown in Table 1.,and they are all different positive qualitative values, less or equal to the input positive qualitative values, except for the case a) when the result is the fuzzy set of zero.

It is important to note that here [1 1 1] is not a representation of the value '?', because we are not in the sign quantity space. After interpretation [1 1 1] corresponds to zero, because

it is obtained as a result of addition of positive and negative qualitative value of similar magnitude.

Table 1. Transformation of ambiguous qualitative values to strictly positive or negative qualitative values

case	a	b	c	d	e	f
U*	[1 1 1]	[0.5 0.5 1]	[0.25 0.25 1]	[0.2 0.2 1]	[0.1 0.2 1]	[0.2 1 0.5]
$\hat{u}$	2	2.25	2.5	2.57	2.69	2.18
W*	[0 1 0]	[0 0.5 1]	[0 0.25 1]	[0 0.2 1]	[0 0.2 1]	[0 1 0.35]

Subtraction and division

Another two important binary algebraic operations are subtraction and division. *Williams* has noticed [8] that the subtraction $\ominus$ could not be defined as the inverse of addition $\oplus$, but $\ominus$ is related to $\otimes$ in a manner similar to ordinary algebra:

$$\ominus \underline{Y}^* = [1\ 1\ 0] \otimes \underline{Y}^* \qquad (27)$$

$$\underline{X}^* \ominus \underline{Y}^* = \underline{X}^* \oplus ([1\ 1\ 0] \otimes \underline{Y}^*) \qquad (28)$$

where [1 1 0] is a fuzzy representation of negative unity (-1). As a result of this the properties

$$\ominus(\ominus \underline{X}^*) = \underline{X}^* \qquad (29)$$

$$\ominus(\underline{X}^* \oplus \underline{Y}^*) = \ominus \underline{X}^* \ominus \underline{Y}^* \qquad (30)$$

are preserved.

Division with qualitative values which include 0 is not allowed. This means that the divisor $\underline{Y}^*$ in $\underline{X}^* \ominus \underline{Y}^*$ must be always strictly positive or negative qualitative value, so one and only one of the elements y_1 or y_3 of the fuzzy vector $\underline{Y}^* = [y_1\ y_2\ y_3]$ must be equal to 0. This is the reason why second approach to interpretation of ambiguous values is more convenient.

If the divisor $\underline{Y}^* = [y_1\ y_2\ y_3]$ is a strictly positive or negative qualitative value, the multiplication inverse $(\underline{Y}^*)^{-1}$ could be define as:

$$(\underline{Y}^*)^{-1} = [y_{I1}\ y_{I2}\ y_{I3}] \qquad (31)$$

where for i = 1,2,3

$$y_{Ii} = \begin{cases} (1/y_i)/y, & \text{if } y_i \neq 0 \\ 0, & \text{if } y_i = 0 \end{cases} \qquad (32)$$

and

$$y = \bigvee_i (1/y_i), \quad \text{for } y_i \neq 0 \qquad (33)$$

In this case the equation

$$\underline{Y}^* \otimes (\underline{Y}^*)^{-1} = [0\ 1\ 1] \qquad (34)$$

is always satisfied. An example is the big positive qualitative value [0 0.2 1] and its inverse [0 1 0.2]. Their product is [0 1 1].

Now division could be defined with the equation

$$\underline{X}^* \oslash \underline{Y}^* = \underline{X}^* \otimes (\underline{Y}^*)^{-1} \qquad (35)$$

The properties as

$$((\underline{Y}^*)^{-1})^{-1} = \underline{Y}^* \qquad (36)$$

or

$$\underline{X}^* \oslash (\underline{Y}^* \otimes \underline{Z}^*) = (\underline{X}^* \oslash \underline{Y}^*) \otimes ([0\ 1\ 1] \oslash \underline{Z}^*) = (\underline{X}^* \oslash \underline{Y}^*) \otimes (\underline{Z}^*)^{-1} \qquad (37)$$

are preserved, but the same as for multiplication the distributivity low is not valid

$$(X^* \oplus Y^*) \oslash Z^* \neq (X^* \oslash Z^*) \oplus (Y^* \oslash Z^*) \qquad (38)$$

except for qualitative values from the sign quantity space.

Operation division satisfies quite good common knowledge laws as for example that the result of division of two positive values bigger than unity must be smaller than unity if the divisor is bigger than dividend

$$[0\ 0.5\ 1] \oslash [0\ 0.2\ 1] = [0\ 1\ 0.4]$$

Order of magnitude reasoning

The proposed method is suitable not only for absolute or relative magnitude reasoning, but also for order of magnitude reasoning [4,5]. Here the method introduced by *Mavrovouniotis and Stephanopoulos* [5] will be shortly analysed. They defined seven primitive order of magnitude relations to relative magnitudes. These are $A \ll B$ (intuitively: A is much smaller than B), $A \prec B$ (A is moderately smaller than B), $A \precsim B$ (A is slightly smaller than B), $A \equiv B$ (A is exactly equal to B) and corresponding duals for the first three relations $A \gg B$, $A \succ B$ and $A \succsim B$. The relation $(A \circledast B)$ is equivalent to $(A/B \circledast 1)$.

This seven primitives expressed in terms of fuzzy sets are shown in Table 2., together with acceptable values of elements x_1 and x_2 from the fuzzy vector $A \oslash B = [0\ x_1\ x_2]$ which is the fuzzy set representation of A/B.

Table 2. Primitive relations of the order of magnitude reasoning expressed in terms of fuzzy sets

Relation	Fuzzy set representation $A \oslash B = [0\ x_1\ x_2]$	Intervals of acceptable values x_1	x_2
$A \gg B$	[0 0.2 1]	[0 , 0.4]	1
$A \succ B$	[0 0.5 1]	(0.4 , 0.6]	1
$A \succsim B$	[0 0.8 1]	(0.6 , 1)	1
$A \equiv B$	[0 1 1]	1	1
$A \precsim B$	[0 1 0.8]	1	(0.6 , 1)
$A \prec B$	[0 1 0.5]	1	(0.4 , 0.6]
$A \ll B$	[0 1 0.2]	1	[0 , 0.4]

Primitive relations defined with Table 2. satisfy many common sense inferences. Typical example is

$$A \succsim B \ \&\ B \succsim C \rightarrow A \succsim C$$

$$A \oslash B = [0\ 0.8\ 1]$$

$$B \oslash C = [0\ 0.8\ 1]$$

$$A \oslash C = (A \oslash B) \otimes (B \oslash C) = [0\ 0.8\ 1] \otimes [0\ 0.8\ 1] = [0\ 0.64\ 1]$$

and from Table 2. we can see that [0 0.64 1] belongs to $A \succsim C$.

3. EXAMPLE OF REASONING

Qualitative reasoning with more complicated quantity space proposed in this paper could be used for many qualitative reasoning tasks as for example for qualitative structural model identification, analyses and synthesis [7], qualitative prediction of system behavior [1], diagnostics [9] etc. Here a short example with relative magnitude reasoning will be described just to illustrate the procedure.

Let us suppose that we have a two variable stable system described with equation

$$dx_i/dt = f_i(x_1, x_2, p_1, p_2, t) \qquad (39)$$

where x_i are dependant variables, p_i are independent parameters and f_i is unknown, generally non-linear function. When the system is in steady-state condition all variables have certain equilibrium level, but if independent parameters change the rate of change of variables (dx_i/dt)

will be also changed and as a consequence variables will have new equilibrium levels. The task is to find are these new equilibria higher, lower or the same as previously if only qualitative information about variables interactions are known.

To find how variables will change partial derivates of (39) with respect to parameter p_k have to be equated to 0. For both variables the matrix equation is

$$(\partial \underline{f}/\partial p_k) + \underline{J} \cdot (\partial \underline{x}/\partial p_k) = 0 \qquad (40)$$

where

$$(\partial \underline{f}/\partial p_k) = [\partial f_1/\partial p_k \ \ \partial f_2/\partial p_k]^T$$

$$(\partial \underline{x}/\partial p_k) = [\partial x_1/\partial p_k \ \ \partial x_2/\partial p_k]^T$$

and $\underline{J}$ is a Jacobian matrix in steady state condition

$$\underline{J} = [\partial \underline{f}/\partial \underline{x}] = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

The task is to find $(\partial \underline{x}/\partial p_k)$ if qualitative information about $(\partial \underline{f}/\partial p_k)$ and $\underline{J}$ are only avaible.

Let us suppose that we know that the influence of x_2 to $\partial x_1/\partial t$ is negative (sgn x_{12} = -) that both influences of x_1 to $\partial x_1/\partial t$ and to $\partial x_2/\partial t$ are positive (sgn x_{11} = sgn x_{21} = +) and that x_2 has no influence at all on $\partial x_2/\partial t$ (sgn x_{22} = 0). Additionally we know that x_{12} is the biggest according to its magnitude and that is much bigger than x_{21} and moderately bigger than x_{11}. This situation could be represented in two different ways: using the smallest or the biggest element according to magnitude as a representative element and than compare all other elements with them. In both cases the Jacobian matrix is the matrix of fuzzy vectors, but in the first case all fuzzy vectors are bigger or equal to unity (positive or negative), and in the second case they are smaller or equal to unity. Here the second case will be used, so the Jacobian matrix is

$$\underline{J} = \begin{bmatrix} [0\ 1\ 0.5] & [1\ 1\ 0] \\ [0\ 1\ 0.2] & [0\ 1\ 0] \end{bmatrix}$$

The question is : If $\partial x_1/\partial t$ and $\partial x_2/\partial t$ change the same (caused by some, not necessary known, parameters change) and their change is, let us say positive big [0 0.2 1], how will the variable x_2 behave ? Which will have a bigger influence on the change of the variable x_2 equilibrium level ? This is an example of the ambiguous case which is unsolvable for qualitative values from the sign quantity space Q.

To solve this problem the element $\partial x_2/\partial p_k$ of the vector $(\partial \underline{x}/\partial p_k)$ has to be calculated if

$$\text{a) } (\partial \underline{f}/\partial p_1) = \begin{bmatrix} [0\ 0.2\ 1] \\ [0\ 1\ 0] \end{bmatrix}, \text{ and b) } (\partial \underline{f}/\partial p_2) = \begin{bmatrix} [0\ 1\ 0] \\ [0\ 0.2\ 1] \end{bmatrix}$$

The case a) corresponds to the change of $\partial x_1/\partial t$ and the case b) to the of change of $\partial x_2/\partial t$. In the first case the matrix equation is

$$\begin{bmatrix} [0\ 1\ 0.5] & [1\ 1\ 0] \\ [0\ 1\ 0.2] & [0\ 1\ 0] \end{bmatrix} \otimes \begin{bmatrix} [\partial x_1/\partial p_1] \\ [\partial x_2/\partial p_1] \end{bmatrix} = [1\ 1\ 0] \otimes \begin{bmatrix} [0\ 0.2\ 1] \\ [0\ 1\ 0] \end{bmatrix}$$

The system determinant and the determinant of the second unknown are

$$D = [0\ 1\ 0.5] \otimes [0\ 1\ 0] \ominus [1\ 1\ 0] \otimes [0\ 1\ 0.2] = [0\ 0.2\ 1]$$

$$D2 = \ominus([1\ 0.2\ 0] \otimes [0\ 1\ 0.2]) = [0\ 1\ 1]$$

so $\quad [\partial x_2/\partial p_1] = [0\ 1\ 1] \oslash [0\ 0.2\ 1] = [0\ 1\ 0.2]$

In the case b) D is the same and D2 = [1 0.4 0], so

$$[\partial x_2/\partial p_2] = [1\ 0.4\ 0] \oslash [0\ 0.2\ 1] = [0.5\ 1\ 0]$$

The conclusion is that x_2 will have higher new equilibrium level under influence of $\partial x_1/\partial t$ change, and lower new equilibrium level under influence of $\partial x_2/\partial t$ change. If we compare the magnitudes we can see that the magnitude in the first case is smaller than the magnitude in the second case, so final conclusion is that if both $\partial x_1/\partial t$ and $\partial x_2/\partial t$ change the same the new equilibrium level of the variable x_2 will be lower than the present equilibrium level . It is possible even to compare magnitudes of x_2 change, first multiplying $[\partial x_2/\partial p_2]$ with [1 1 0], because it is negative. The result is

$$[\partial x_2/\partial p_1] \oslash ([1\ 1\ 0] \otimes [\partial x_2/\partial p_2]) = [0\ 1\ 0.4]$$

Using the Table 2. the conclusion could be that the magnitude of change of x_2 due to $\partial x_1/\partial t$ change is much smaller than due to $\partial x_2/\partial t$ change.

4. CONCLUSION

Qualitative reasoning is very important in everyday life, so it is not surprising that it has been intensively studied. The simplest quantity spaces that could be used in qualitative reasoning is the sign quantity space Q = {+,-,0,?}. One of the weak points of reasoning with this quantity space is its inherent ambiguity in connection with the addition of positive and negative values. This ambiguity could be resolved only if at least partial knowledge about variable magnitudes is known.

The paper describes how this knowledge, which is in lot of cases vague and subjective, could be mathematically represented and used in qualitative reasoning. The approach is based on the theory of fuzzy sets. The variable qualitative values are modeled with fuzzy sets and algebraic operations are defined in terms of manipulation with fuzzy sets membership functions.

It is important to stress that the reasons why the fuzzy set theory was introduced 25 years ago were very similar to the reasons behind the introduction of qualitative reasoning theory, so it was natural to try to connect these two theories, and that was the main aim of this paper. With this paper we want to show how this two theories , both suitable to deal with qualitative information, could be combined and used together.

The obtained results are quite satisfactory and encouraging. Although it was here shown only how the qualitative reasoning could have benefits from the fuzzy set theory, we truly belive that the fuzzy set theory could learn a lot from the qualitative reasoning, too, specially to tackle problems which include plant (organisational) complexity [10,11]. We hope that in future it will be more research aimed to connect both theories even more deeper.

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21/5