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# QUANTITY SPACES IN QUALITATIVE REASONING

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*The paper deals with mathematical properties of quantity spaces used in qualitative reasoning. The first and most important one is a sign quantity space with four qualitative values: positive,negative,zero and ambivalent, which has been used in lot of qualitative reasoning applications. To resolve inherent ambiguity of sign quantity spaces, more complicated quantity spaces must be introduced. Here few approaches based on ordinal representation are discussed. At the end, as a concluding remarks, some observations about philosophical background of sign quantity space are given.*

KEY WORDS *sign algebra/ordinal representation/quaternity*

## 1. INTRODUCTION

A model as an intellectual construct is very important in human reasoning about the world. One of the reasons why human behavior is (relatively) successful is certainly the propensity of the human brain to create qualitative models of the world and to use them in qualitative reasoning about it. As De Kleer and Brown *(De Kleer,Brown, 1983)* said

"*...humans appear to use a qualitative causal calculus in reasoning about the behavior of their physical environment*".

Because of that it is not surprising that lot of attempts have been made, especially in the last decade, to understand and to apply qualitative reasoning to different areas of human scientific thinking.

A lot of approaches to qualitative reasoning have been established *(Bredeweg,Wielinga,1988)*and in each of them different qualitative values of parameters have been used. These qualitative values belong to different quantity spaces. In this paper we have analysed mathematical properties of one of this spaces. That is maybe the most important one, and certainly the most used one: sign quantity space. Also we have discussed some more complicated quantity spaces created by introducing new landmarks on the real line. At the end some observations about philosophical background of sign quantity space are given.

## 2. SIGN QUANTITY SPACE

In 1946. Heider *(Heider,1946)* used quantity space with three values: positive, negative and zero to analyse complex psychological behavior of people in the small group. After him, qualitative reasoning with this sign values has found a lot of others interesting and useful applications.

In 1964. Harrary et al. *(Harrary et al,1964)* introduced sign arithmetics as a suitable mathematical approach to the calculation of the balance of sign structural models. Instead of a three values quantity space {+,−,0,?} they have used four values quantity space Q = {+,−,0,?} where ? means ambivalent, ambiguous and express the lack of knowledge about exact sign values. They defined elementary arithmetic operations addition and multiplication with Cayley tables using common sense about calculation with signs.

| ⊕ | 0 | + | − | ? |
|---|---|---|---|---|
| 0 | 0 | + | − | ? |
| + | + | + | ? | ? |
| − | − | ? | − | ? |
| ? | ? | ? | ? | ? |

(1)

| ⊗ | 0 | + | − | ? |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| + | 0 | + | − | ? |
| − | 0 | − | + | ? |
| ? | 0 | ? | ? | ? |

(2)

Recently Struss *(Struss,1988)* introduced interval arithmetic as a more general form than sign arithmetic. He defined elementary algebraic operations in terms of the well known interval manipulations. The core of his approach is the representation of sign values with intervals, so the quantity space Q is a set of open and closed intervals defined with a set of landmarks on the real line.

$$Q = \{ (a,b) \quad a,b \in B \} \cup \{[a,a] \quad a \in B\{-\infty,\infty\}\} \qquad (3)$$

For sign quantity space the set of landmarks is $B = \{-\infty,\infty\}$, so + correspond to open interval $(0,\infty)$, − to $(-\infty,0)$, ? to $(-\infty,\infty)$ and 0 to the close interval $[0,0]$.

Struss found that algebraic structure (Q,⊕) is a commutative semigroup with an identity element (additional identity), namely [0,0], but he didn't analyse the property of (Q,⊕,⊗), and this will be done in this chapter.

Algebraic structure (Q,⊕,⊗), where Q = {+,−,0,?} and operation ⊕ and ⊗ are defined with Cayley tables (1) and (2), is a sign algebra and it belongs to a group of path algebras, because:

a) operation ⊕ is associative, commutative and idempotent,

b) operation ⊗ is associative and commutative,

c) both (Q,⊕) and (Q,⊗) have an identity element and a zero element and the additional identity is equal to the multiplicative zero, and

d) ⊗ is distributive over ⊕.

Proofs can be found in *(Stipaničev,1989)*. The additional identity and the multiplicative zero is the element 0, the additional zero is the element ? and the multiplicative identity is the element +.

Since the operation ⊕ in Q is idempotent, commutative and associative, it is possible to define an ordering ≤ of Q by the rule

$$\forall\, x,y \in Q\,,\quad x \le y\,,\ \text{if and only if}\ \ x \oplus y = y \qquad (4)$$

A Hasse diagram of the set Q is shown in Fig.1

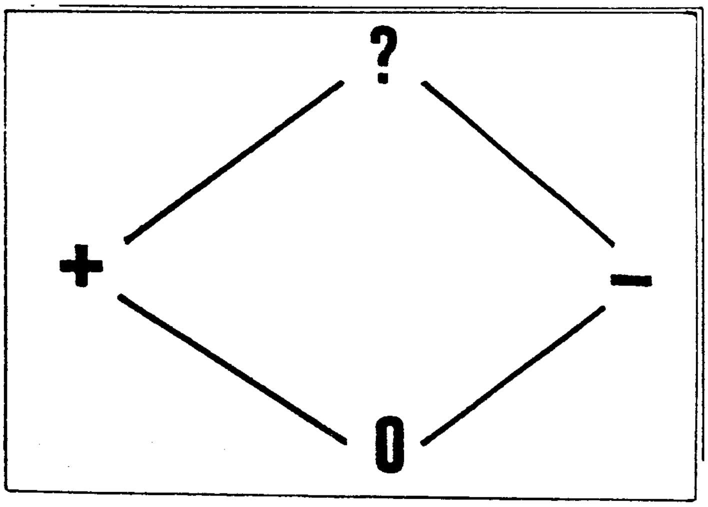


Fig.1. Hasse diagram of sign quantity space Q according to ⊕

Sign quantity space Q is ordered, but not totally ordered, because + and – are not comparable. (Q,⊕) is a semilattice, but (Q,⊗) is not, because ⊗ is not idempotent, so (Q,⊕,⊗) can not form a lattice.

These mathematical properties of sign quantity space are quite important for qualitative reasoning with sign values, as for example for prediction of ambiguous cases in the analyses of algebraic sign equation solvability *(Stipaničev,1989).*

## 4. MORE COMPLICATED QUANTITY SPACES

One of disadvantages of qualitative reasoning with sign values and sign quantity space is its inherent ambiguity. In lot of qualitative inferences the obtained result is ?, what means that the exact answer could not be find. To solve this problem, one of approaches is to introduce more complicated quantity space with more qualitative values. This could be done by either *ordinal representation*, introducing, beside landmarks $-\infty$, 0 and $\infty$, new landmarks on the real line, or by *categorical representation* which use separate sets of simple symbolic values such as {low, medium, high} or similar. These values are defined by reference to real world values. Here only ordinal representation will be discussed.

The easiest way to introduce new landmarks is to define them as characteristic points, as for example local minimums, local maximums or inflection points. The landmark values form a partially ordered set B, and the quantity space could also be defined as a set of open and closed intervals introduced by (3).

But if new landmarks, beside $-\infty$, 0 and $\infty$ are introduced, many new problems could arise *(Struss,1988)*, as for example finiteness of quantity space, the problem with definition of operation subtraction, etc. Some of them could be resolved by restricting the quantity space to a set of elementary intervals and modifying the operation addition, but then other problems occur, typically the operation addition is not any more associative.

Shen and Leitch *(Shen,Leitch,1989)* proposed another approach, defining landmarks as intervals and qualitative values as fuzzy numbers. Fuzzy numbers can be generally represented with four values q = (a,b,u,v) where a and b are landmarks which define the flat part of the fuzzy number and u and v are left and right spreads of the fuzzy number. Intervals are special case of fuzzy numbers where u = v = 0, and the same is for the single points where u = v = 0 and a = b.Algebraic operations are defined as usually in the theory of fuzzy sets using Zadeh extension principle. This approach is more flexible, but many problems caused by introducing landmarks are still present, typically addition is not associative or operation division can not be easily defined.

To avoid these problems we have proposed a different approach, also based on fuzzy set theory *(Stipaničev,Efstathiou,1989)*. The core of our approach is to define qualitative values with three element fuzzy vector whose support set is the same as the set of landmarks of simple sign quantity space B = {-∞,∞}. This approach has many advantages. It satisfies a lot of

a) *formal arithmetic requirements*, as for example associativity, commutativity, etc.

b) *common knowledge requirements*, and this means that our qualitative arithmetic coincide quite well with qualitative interpretation of real line arithmetics and with human common sense, and

c) *implementation requirements*, because computation procedures are simple, interpretation and presentation of results easy and similar.

The value of our fuzzy set membership function could be seen as a plausibility degree to which the qualitative value is more or less "close" to the appropriate landmark. For example the membership function of the big positive value could be expressed, using the same notation from the fuzzy set theory, as 0/-∞ + 0.2/0 + 1/∞ . Note that + is not addition but union of fuzzy singletons, eg. fuzzy sets whose support set has only one element).

More convenient representation is with fuzzy vector [ 0 0.2 1 ], where the first element correspond to the value of membership function of the support element -∞, the second element to the membership value of the support element 0 and the third element to the support element ∞. Similarly a positive qualitative value with smaller magnitude could be expressed with [ 0 0.8 1 ], because now it is more close to the landmark 0, so the value of the second element is bigger. Negative value with similar magnitude could be expressed with [ 1 0.8 0 ]. It is important to stress that assignment of membership values is a subjective, but not arbitrary process.

Results obtained with our approach are satisfactory and encouraging. They correspond quite well to our common knowledge and qualitative interpretation of arithmetics on the real line. Typical example is that the result of multiplication of the value bigger than one and value smaller than one must be somewhere between them. Using this approach it is possible to perform quite easily qualitative reasoning with order of magnitudes, too.

## 5. CONCLUDING REMARKS

In coping with real world problems and situations there is no doubt that humans prefer to use qualitative information instead of quantitative one. Reasoning with qualitative values, specially in constructing behavioral, causal models of the world is one important feature of humans thinking.

Sign values from sign quantity space have special place in this activity. Negative and positive form a structure called *dyad* and dyads have been very important in philosophical thinking about world from Greek times till today. It is not necessary to imagine positive and negative just as a subset of real numbers. They are also opposite concepts, opposite features, opposite values or opposite behavior, as for example love and hate, good and bad. Adding the third value zero which express the absence of both concepts, which represent the synthesis point between opposite meanings, the *triad* is constructed. But as Plato said in Timaeus: *"One, two, three - but ... where is the forth?"* The forth is ambiguity which represents the absence of knowledge, which incorporate all previous concepts. A new structure called *tetrad* or in Jung's expression *quaternity* is obtained and as Jung said *(Jung, 1958):*

> *"The quaternity is an archetype of almost universal occurrence. It forms the logical bases for any whole judgment."*

It is interesting that the formal mathematical organisation of sign quantity space, after the introduction of ordering is in a quaternity structure as the Hasse diagram in Fig.1 shows. So, possibly that is the reason why sign quantity space is so important either in human reasoning and artificial qualitative reasoning.

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